ECE 604, Lecture 39

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1 Quantum Coherent State of Light

We have seen that a photon number state[1] of a quantum pendulum do not have a classical correspondence as the average or expectation values of the position and momentum of the pendulum are always zero for all time for this state. Therefore, we have to seek a time-dependent quantum state that has the classical equivalence of a pendulum. This is the coherent state, which is the contribution of many researchers, most notably, George Sudarshan (1931–2018) and Roy Glauber (born 1925) in 1963. Glauber was awarded the Nobel prize in 2005.

We like to emphasize again that the mode of an electromagnetic oscillation is homomorphic to the oscillation of classical pendulum. Hence, we first connect the oscillation of a quantum pendulum to a classical pendulum. Then we can connect the oscillation of a quantum electromagnetic mode to the classical electromagnetic mode and then to the quantum pendulum.

1.1 Quantum Harmonic Oscillator Revisited

To this end, we revisit the quantum harmonic oscillator or the quantum pendulum with more mathematical depth. Rewriting the eigen-equation for the photon number state for the quantum harmonic oscillator, we have

$$\hat{H}\psi_n(x) = \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega_0^2 x^2\right]\psi_n(x) = E_n\psi_n(x). \tag{1.1}$$

where $\psi_n(x)$ is the eigenfunction, and E_n is the eigenvalue. The above can be changed into a dimensionless form first by dividing $\hbar\omega_0$, and then let $\xi = \sqrt{\frac{m\omega_0}{\hbar}}x$ be a dimensionless variable. The above then becomes

$$\frac{1}{2}\left(-\frac{d^2}{d\xi^2} + \xi^2\right)\psi(\xi) = \frac{E}{\hbar\omega_0}\psi(\xi) \tag{1.2}$$

We can define $\hat{\pi} = -i\frac{d}{d\xi}$ and $\hat{\xi} = \hat{I}\xi$ to rewrite the Hamiltonian as

$$\hat{H} = \frac{1}{2}\hbar\omega_0(\hat{\pi}^2 + \hat{\xi}^2) \tag{1.3}$$

Furthermore, the Hamiltonian in (1.1) looks almost like $A^2 - B^2$, and hence motivates its factorization. To this end, it can be rewritten

$$\frac{1}{\sqrt{2}}\left(-\frac{d}{d\xi} + \xi\right)\frac{1}{\sqrt{2}}\left(\frac{d}{d\xi} + \xi\right) = \frac{1}{2}\left(\frac{-d^2}{d\xi^2} + \xi^2\right) - \frac{1}{2}\left(\frac{d}{d\xi}\xi - \xi\frac{d}{d\xi}\right) \tag{1.4}$$

It can be shown easily that as operators (meaning that they will act on a function to their right),

$$\left(\frac{d}{d\xi}\xi - \xi\frac{d}{d\xi}\right) = \hat{I} \tag{1.5}$$

[1] In quantum theory, a "state" is synonymous with a state vector or a function.

Therefore

$$\frac{1}{2}\left(-\frac{d^2}{d\xi^2}+\xi^2\right)=\frac{1}{\sqrt{2}}\left(-\frac{d}{d\xi}+\xi\right)\frac{1}{\sqrt{2}}\left(\frac{d}{d\xi}+\xi\right)+\frac{1}{2} \tag{1.6}$$

We define the operator

$$\hat{a}^\dagger=\frac{1}{\sqrt{2}}\left(-\frac{d}{d\xi}+\xi\right) \tag{1.7}$$

The above is the creations, or raising operator and the reason for its name is obviated later. Moreover, we define

$$\hat{a}=\frac{1}{\sqrt{2}}\left(\frac{d}{d\xi}+\xi\right) \tag{1.8}$$

which represents the annihilation or lowering operator. With the above definitions of the raising and lowering operators, it is easy to show that by straightforward substitution that

$$\left[\hat{a},\hat{a}^\dagger\right]=\hat{I} \tag{1.9}$$

Therefore, Schrödinger equation for quantum harmonic oscillator can be rewritten more concisely as

$$\frac{1}{2}\left(\hat{a}^\dagger\hat{a}+\hat{a}\hat{a}^\dagger\right)\psi=\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right)\psi=\frac{E}{\hbar\omega_0}\psi \tag{1.10}$$

In mathematics, a function is analogous to a vector. So ψ is the implicit representation of a vector. The operator

$$\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right)$$

is an implicit[2] representation of an operator, and in this case, a differential operator. So the above, (1.10), is analogous to the matrix eigenvalue equation $\overline{\mathbf{A}}\cdot\mathbf{x}=\lambda\mathbf{x}$.

Consequently, the Hamiltonian operator can be expressed concisely as

$$\hat{H}=\hbar\omega_0\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right) \tag{1.11}$$

Equation (1.10) above is in implicit math notation. In implicit Dirac notation, it is

$$\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right)|\psi\rangle=\frac{E}{\hbar\omega_0}|\psi\rangle \tag{1.12}$$

[2]A notation like $\overline{\mathbf{A}}\cdot\mathbf{x}$, we will call implicit, while a notation $\sum_{i,j}A_{ij}x_j$, we will call explicit.

In the above, $\psi(\xi)$ is a function which is a vector in a functional space. It is denoted as ψ in math notation and $|\psi\rangle$ in Dirac notation. This is also known as the "ket". The conjugate transpose of a vector in Dirac notation is called a "bra" which is denoted as $\langle\psi|$. Hence, the inner product between two vectors is denoted as $\langle\psi_1|\psi_2\rangle$ in Dirac notation.[3]

If we denote a photon number state by $\psi_n(x)$ in explicit notation, ψ_n in math notation or $|\psi_n\rangle$ in Dirac notation, then we have

$$\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)|\psi_n\rangle = \frac{E_n}{\hbar\omega_0}|\psi_n\rangle = \left(n + \frac{1}{2}\right)|\psi_n\rangle \tag{1.13}$$

where we have used the fact that $E_n = (n + 1/2)\hbar\omega_0$. Therefore, by comparing terms in the above, we have

$$\hat{a}^\dagger\hat{a}|\psi_n\rangle = n|\psi_n\rangle \tag{1.14}$$

and the operator $\hat{a}^\dagger\hat{a}$ is also known as the number operator because of the above. It is often denoted as

$$\hat{n} = \hat{a}^\dagger\hat{a} \tag{1.15}$$

It can be further shown by direct substitution that

$$\hat{a}|\psi_n\rangle = \sqrt{n}|\psi_{n-1}\rangle \Leftrightarrow \hat{a}|n\rangle = \sqrt{n}|n-1\rangle \tag{1.16}$$

$$\hat{a}^\dagger|\psi_n\rangle = \sqrt{n+1}|\psi_{n+1}\rangle \Leftrightarrow \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \tag{1.17}$$

hence their names on lowering and raising operator.[4]

2 Some Words on Quantum Randomness and Quantum Observables

We saw previously that in classical mechanics, the conjugate variables p and x are deterministic variables. But in the quantum world, they become random variables. It was quite easy to see that x is a random variable in the quantum world. But the momentum p is elevated to become a differential operator $\hat{p}$, and it is not clear that it is a random variable anymore. But we found its expectation value nevertheless in the previous lecture.

It turns out that we have to extend the concept of the average of a random variable to taking the "average" of an operator, which is the elevated form of a random variable. Now that we know Dirac notation, we can write the expectation value of the operator $\hat{p}$ with respect to a quantum state ψ as

$$\langle p\rangle = \langle\psi|\hat{p}|\psi\rangle \tag{2.1}$$

[3]There is a one-to-one correspondence of Dirac notation to matrix algebra notation. $\overline{\mathbf{A}}\cdot\mathbf{x} \leftrightarrow \hat{A}|x\rangle$, $\langle x| \leftrightarrow \mathbf{x}^\dagger$ $\langle x_1|x_2\rangle \leftrightarrow \mathbf{x}_1^\dagger\cdot\mathbf{x}_2$.

[4]The above notation for a vector could appear cryptic or too terse to the uninitiated. To parse it, one can always down-convert from an abstract notation to a more explicit notation. Namely $|n\rangle \rightarrow |\psi_n\rangle \rightarrow \psi_n(\xi)$.

The above is the elevated way of taking the "average" of an operator. As mentioned before, Dirac notation is homomorphic to matrix algebra notation. The above is similar to $\boldsymbol{\psi}^\dagger \cdot \overline{\mathbf{P}} \cdot \boldsymbol{\psi}$. This quantity is always real if $\overline{\mathbf{P}}$ is a Hermitian matrix. Hence, in (2.1), the expectation value is always real if $\hat{p}$ is Hermitian. In fact, it can be proved the it is Hermitian in the function space that it is defined.

Operators that correspond to measurable quantities are called observables in quantum theory, and they are replaced by operators in the quantum world. We can take expectation values of these operators with respect to the quantum state involved. Therefore, these observables will have a mean and standard deviation. In the previous section, we elevated the position variable ξ to become an operator $\hat{\xi} = \xi \hat{I}$. This operator is clearly Hermitian, and hence, the expectation value of this position operator is always real. From the previous section, we see that the normalized momentum operator is always Hermitian, and hence, its expectation value is always real. The difference of these quantum observables compared to classical variables is that the quantum observables have a mean and a standard deviation just like a random variable.

3 Derivation of the Coherent States

As one cannot see the classical pendulum emerging from the photon number states, one needs another way of bridging the quantum world with classical world. This is the role of the coherent state: It will show the correspondence principle, and that a classical pendulum does emerge from a quantum pendulum when the energy of the pendulum is large. Hence, it will be interesting to see how the coherent state is derived. The derivation of the coherent state is more math than physics. Nevertheless, the derivation is interesting. We are going to present it according to the simplest way presented in the literature. There are deeper mathematical methods to derive this coherent state like Bogoliubov transform which is outside the scope of this course.

Now, endowed with the needed mathematical tools, we can derive the coherent state. To say succinctly, the coherent state is the eigenstate of the annihilation operator, namely that

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \tag{3.1}$$

Here, we use α as an eigenvalue as well as an index or identifier of the state $|\alpha\rangle$.[5] Since the number state $|n\rangle$ is complete, the coherent state $|\alpha\rangle$ can be expanded in terms of the number state $|n\rangle$. Or that

$$|\alpha\rangle = \sum_{n=0}^{\infty} C_n |n\rangle \tag{3.2}$$

[5]This notation is cryptic and terse, but one can always down-convert it as $|\alpha\rangle \rightarrow |f_\alpha\rangle \rightarrow f_\alpha(\xi)$ to get a more explicit notation.

When the annihilation operator is applied to the above, we have

$$\hat{a}|\alpha\rangle = \sum_{n=0}^{\infty} C_n \hat{a}|n\rangle = \sum_{n=1}^{\infty} C_n \hat{a}|n\rangle = \sum_{n=1}^{\infty} C_n \sqrt{n}|n-1\rangle$$
$$= \sum_{n=0}^{\infty} C_{n+1}\sqrt{n+1}|n\rangle \tag{3.3}$$

Equating the above with $\alpha|\alpha\rangle$, then

$$\sum_{n=0}^{\infty} C_{n+1}\sqrt{n+1}|n\rangle = \alpha \sum_{n=0}^{\infty} C_n|n\rangle \tag{3.4}$$

By the orthonormality of the number states $|n\rangle$, then we can take the inner product of the above with $\langle n|$ and making use of the orthonormal relation that $\langle n'|n\rangle = \delta_{n'n}$ to remove the summation sign. Then we arrive at

$$C_{n+1} = \alpha C_n/\sqrt{n+1} \tag{3.5}$$

Or recursively

$$C_n = C_{n-1}\alpha/\sqrt{n} = C_{n-2}\alpha^2/\sqrt{n(n-1)} = \ldots = C_0\alpha^n/\sqrt{n!} \tag{3.6}$$

Consequently

$$|\alpha\rangle = C_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{3.7}$$

But due to the probabilistic interpretation of quantum mechanics, the state vector $|\alpha\rangle$ is normalized to one, or that[6]

$$\langle\alpha|\alpha\rangle = 1 \tag{3.8}$$

Then

$$\langle\alpha|\alpha\rangle = C_0^* C_0 \sum_{n,n'}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \frac{\alpha^{n'}}{\sqrt{n'!}} \langle n'|n\rangle$$
$$= |C_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} = |C_0|^2 e^{|\alpha|^2} = 1 \tag{3.9}$$

Therefore, $C_0 = e^{-|\alpha|^2/2}$, or that

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{3.10}$$

[6]The expression can be written more explicitly as $\langle\alpha|\alpha\rangle = \langle f_\alpha|f_\alpha\rangle = \int_{\infty}^{\infty} d\xi f_\alpha^*(\xi) f_\alpha(\xi) = 1$.

In the above, to reduce the double summations into a single summation, we have make use of $\langle n'|n\rangle = \delta_{n'n}$, or that the photon-number states are orthonormal. Also since $\hat{a}$ is not a Hermitian operator, its eigenvalue α can be a complex number.

Since the coherent state is a linear superposition of the photon number states, the average number of photons can be associated with the coherent state. If the average number of photons embedded in a coherent is N, then $N = |\alpha|^2$. As shall be shown, α is related to the amplitude of the quantum oscillation. The more photons there are, the larger is $|\alpha|$.

3.1 Time Evolution of a Quantum State

The Schrodinger equation can be written concisely as

$$\hat{H}|\psi\rangle = i\hbar\partial_t|\psi\rangle \tag{3.11}$$

The above not only entails the form of Schrodinger equation, it is the form of the general quantum state equation. Since $\hat{H}$ is time independent, the formal solution to the above is

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle \tag{3.12}$$

Applying this to the photon number state with $\hat{H}$ being that of the quantum pendulum, then

$$e^{-i\hat{H}t/\hbar}|n\rangle = e^{-i\omega_n t}|n\rangle \tag{3.13}$$

where $\omega_n = \left(n + \frac{1}{2}\right)\omega_0$. The above simplification follows because $|n\rangle$ an eigenstate of the Hamiltonian $\hat{H}$ for the quantum pendulum.

3.1.1 Time Evolution of the Coherent State

Using the above time-evolution operator, then the time dependent coherent state becomes[7]

$$|\alpha, t\rangle = e^{-i\hat{H}t/\hbar}|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n e^{-i\omega_n t}}{\sqrt{n!}}|n\rangle \tag{3.14}$$

By letting $\omega_n = \omega_0\left(n + \frac{1}{2}\right)$, the above can be written as

$$|\alpha, t\rangle = e^{-i\omega_0 t/2}e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\left(\alpha e^{-i\omega_0 t}\right)^n}{\sqrt{n!}}|n\rangle \tag{3.15}$$

$$= e^{-i\omega_0 t/2}|\alpha e^{-i\omega_0 t}\rangle = e^{-i\omega_0 t/2}|\tilde{\alpha}\rangle \tag{3.16}$$

where $\tilde{\alpha} = \alpha e^{-i\omega_0 t}$. Now we see that the last factor in (3.15) is similar to the expression for a coherent state in (3.10). Therefore, we can express the above more succinctly by replacing α in (3.10) with $\tilde{\alpha} = \alpha e^{-i\omega_0 t}$ as

$$\hat{a}|\alpha, t\rangle = e^{-i\omega_0 t/2}\left(\alpha e^{-i\omega_0 t}\right)|\alpha e^{-i\omega_0 t}\rangle = \tilde{\alpha}|alpha, t\rangle \tag{3.17}$$

[7]Note that $|\alpha, t\rangle$ is a shorthand for $f_\alpha(\xi, t)$.

Therefore, $|\alpha, t\rangle$ is the eigenfunction of the $\hat{a}$ operator. But now, the eigenvalue of the annihilation operator $\hat{a}$ is now a complex number which is a function of time t.

4 More on the Creation and Annihilation Operator

In order to connect the quantum pendulum to a classical pendulum via the coherent state, we will introduce some new operators. Since

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{d}{d\xi} + \xi\right) \tag{4.1}$$

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\frac{d}{d\xi} + \xi\right) \tag{4.2}$$

We can relate $\hat{a}^\dagger$ and $\hat{a}$ to the momentum operator $\hat{\pi}$ and position operator $\hat{\xi}$ previously defined, or that

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(-i\hat{\pi} + \hat{\xi}\right) \tag{4.3}$$

$$\hat{a} = \frac{1}{\sqrt{2}}\left(i\hat{\pi} + \hat{\xi}\right) \tag{4.4}$$

We also notice that

$$\hat{\xi} = \frac{1}{\sqrt{2}}\left(\hat{a}^\dagger + \hat{a}\right) = \xi\hat{I} \tag{4.5}$$

$$\hat{\pi} = \frac{i}{\sqrt{2}}\left(\hat{a}^\dagger - \hat{a}\right) = -i\frac{d}{d\xi} \tag{4.6}$$

Notice that both $\hat{\xi}$ and $\hat{\pi}$ are Hermitian operators in the above, and hence, their expectation values are real. With this, the average or expectation position of the pendulum in normalized coordinate, ξ, can be found by taking expectation with respect to the coherent state, or

$$\langle\alpha|\hat{\xi}|\alpha\rangle = \frac{1}{\sqrt{2}}\langle\alpha|\hat{a}^\dagger + \hat{a}|\alpha\rangle \tag{4.7}$$

Since by taking the complex conjugation transpose of (3.1)[8]

$$\langle\alpha|\hat{a}^\dagger = \langle\alpha|\alpha^* \tag{4.8}$$

and (4.7) becomes

$$\langle\xi\rangle = \langle\alpha|\hat{\xi}|\alpha\rangle = \frac{1}{\sqrt{2}}\left(\alpha^* + \alpha\right)\langle\alpha|\alpha\rangle = \sqrt{2}\Re e[\alpha] \neq 0 \tag{4.9}$$

[8]Dirac notation is homomorphic with matrix algebra notation. $(\overline{\mathbf{a}} \cdot \mathbf{x})^\dagger = \mathbf{x}^\dagger \cdot (\overline{\mathbf{a}})^\dagger$.

Repeating the exercise for time-dependent case, when we let $\alpha \rightarrow \tilde{\alpha}(t) = \alpha e^{-i\omega_0 t}$, then, letting $\alpha = |\alpha| e^{-i\psi}$,

$$\langle \xi(t) \rangle = \sqrt{2}|\alpha| \cos(\omega_0 t + \psi) \tag{4.10}$$

By the same token,

$$\langle P \rangle = \langle \alpha | \hat{P} | \alpha \rangle = \frac{i}{\sqrt{2}} \left(\alpha^* - \alpha \right) \langle \alpha | \alpha \rangle = \sqrt{2} \Im m[\alpha] \neq 0 \tag{4.11}$$

For the time-dependent case, we let $\alpha \rightarrow \tilde{\alpha}(t) = \alpha e^{-i\omega_0 t}$,

$$\langle P(t) \rangle = -\sqrt{2}|\alpha| \sin(\omega_0 t + \psi) \tag{4.12}$$

Hence, we see that the expectation values of the normalized coordinate and momentum just behave like a classical pendulum. There is however a marked difference: These values have standard deviations that are non-zero. Hence, they have quantum fluctuation or quantum noise associated with them. Since the quantum pendulum is homomorphic with the oscillation of a quantum electromagnetic mode, the amplitude of a quantum electromagnetic mode will have a mean and a fluctuation as well.

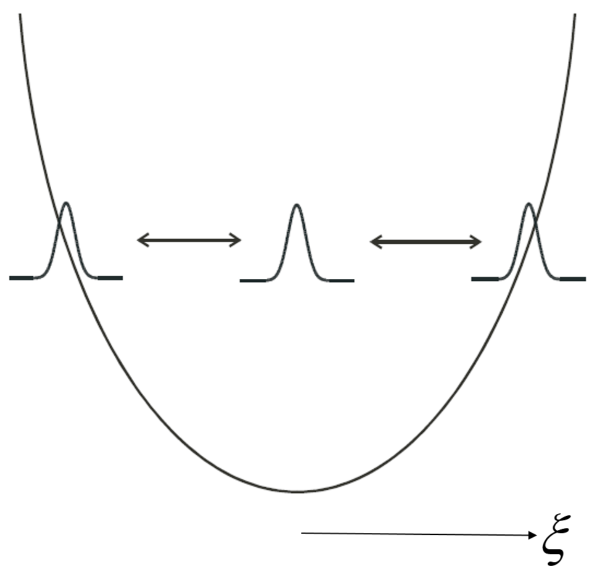

Fig. 3.4. A coherent-state wave function moves through the harmonic oscillator potential, between the classical turning points, without dispersion.

Figure 1: The time evolution of the coherent state. It follows the motion of a classical pendulum or harmonic oscillator (Courtesy of Gerry and Knight).

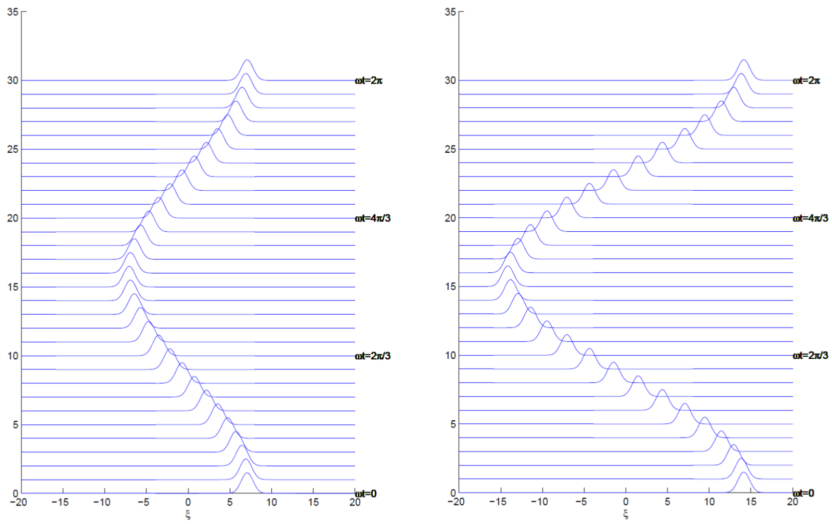

Figure 2: The time evolution of the coherent state for different α's. The left figure is for $\alpha = 5$ while the right figure is for $\alpha = 10$. Recall that $N = |\alpha|^2$.

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4.1 Connecting Quantum Pendulum to Electromagnetic Oscillator

We see that the electromagnetic oscillator in a cavity is similar or homomorphic to a pendulum. We have next to elevate a classical pendulum to become a quantum pendulum. The classical Hamiltonian is

$$H = T + V = \frac{p^2}{2m} + \frac{1}{2} m\omega_0^2 x^2 = \frac{1}{2}\left(P^2(t) + Q^2(t)\right) = E \tag{4.13}$$

In the above, P is a normalized momentum and Q is a normalized coordinate, and their squares have the unit of energy. We have also shown that when the classical pendulum is lifted to be a quantum pendulum, then the quantum Schrodinger equation is

$$\hbar\omega_l \left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right) |\psi, t\rangle = i\hbar\partial_t |\psi, t\rangle \tag{4.14}$$

Our next task is to connect the electromagnetic oscillator to this pendulum. In general, the total energy or the Hamiltonian of an electromagnetic system is

$$H = \frac{1}{2}\int_V d\mathbf{r}\left[\varepsilon \mathbf{E}^2(\mathbf{r}, t) + \frac{1}{\mu}\mathbf{B}^2(\mathbf{r}, t)\right]. \tag{4.15}$$

It is customary to write this Hamiltonian in terms of scalar and vector potentials. For simplicity, we use a 1D cavity, and let $\mathbf{A} = \hat{x}A_x$, $\nabla \cdot \mathbf{A} = 0$ so that $\partial_x A_x = 0$, and letting $\Phi = 0$. Then $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{E} = -\dot{\mathbf{A}}$, and the classical Hamiltonian from (4.15) for a Maxwellian system becomes

$$H = \frac{1}{2} \int\limits_V d\mathbf{r} \left[\varepsilon \dot{\mathbf{A}}^2(\mathbf{r}, t) + \frac{1}{\mu} \left(\nabla \times \mathbf{A}(\mathbf{r}, t) \right)^2 \right]. \tag{4.16}$$

For the 1D case, the above implies that $B_y = \partial_z A_x$, and $E_x = -\partial_t A_x = -\dot{A}_x$. Hence, we let

$$A_x = A_0(t) \sin(k_l z) \tag{4.17}$$

$$E_x = -\dot{A}_0(t) \sin(k_l z) = E_0(t) \sin(k_l z) \tag{4.18}$$

$$B_y = k_l A_0(t) \cos(k_l z). \tag{4.19}$$

where $E_0(t) = -\dot{A}_0(t)$. After integrating over the volume such that $\int_V d\mathbf{r} = \mathcal{A} \int_0^L dz$, the Hamiltonian (4.16) then becomes

$$H = \frac{V_0 \varepsilon}{4} \left(\dot{A}_0(t) \right)^2 + \frac{V_0}{4\mu} k_l^2 A_0^2(t). \tag{4.20}$$

where $V_0 = \mathcal{A}\mathcal{L}$, is the mode volume. The form of (4.20) now has all the elements that make it resemble the pendulum Hamiltonian. We can think of $A_0(t)$ as being related to the displacement of the pendulum. Hence, the second term resembles the potential energy. The first term has the time derivative of $A_0(t)$, and hence, can be connected to the kinetic energy of the system. Therefore, we can rewrite the Hamiltonian as

$$H = \frac{1}{2} \left[P^2(t) + Q^2(t) \right] \tag{4.21}$$

where

$$P(t) = \sqrt{\frac{V_0 \varepsilon}{2}} \dot{A}_0(t) = -\sqrt{\frac{V_0 \varepsilon}{2}} E_0(t), \qquad Q(t) = \sqrt{\frac{V_0}{2\mu}} k_l A_0(t) \tag{4.22}$$

By letting

$$P(t) \to \hat{P} = \sqrt{\hbar \omega_l} \hat{\pi}(t), \qquad Q(t) \to \hat{Q} = \sqrt{\hbar \omega_l} \hat{\xi}(t) \tag{4.23}$$

so that the quantum Hamiltonian now is

$$\hat{H} = \frac{1}{2} \left[\hat{P}^2 + \hat{Q}^2 \right] = \frac{1}{2} \hbar \omega_l \left(\hat{\pi}^2 + \hat{\xi}^2 \right) \tag{4.24}$$

similar to (1.3) as before except now that the resonant frequency of this mode is ω_l instead of ω_0. An equation of motion for the state of the quantum system can be associated with the quantum Hamiltonian just as in the quantum pendulum case.

We have shown previously that

$$\hat{a}^\dagger + \hat{a} = \sqrt{2}\hat{\xi} \tag{4.25}$$

$$\hat{a}^\dagger - \hat{a} = -\sqrt{2}i\hat{\pi} \tag{4.26}$$

Then we can let

$$\hat{P} = -\sqrt{\frac{V_0\varepsilon}{2}}\hat{E}_0 = \sqrt{\hbar\omega_l}\hat{\pi} \tag{4.27}$$

Finally, we arrive at

$$\hat{E}_0 = -\sqrt{\frac{2\hbar\omega_l}{\varepsilon V_0}}\hat{\pi} = \frac{1}{i}\sqrt{\frac{\hbar\omega_l}{\varepsilon V_0}}\left(\hat{a}^\dagger - \hat{a}\right) \tag{4.28}$$

Now that E_0 has been elevated to be a quantum operator $\hat{E}_0$, from (4.18), we can put in the space dependence to get

$$\hat{E}_x(z) = \hat{E}_0 \sin(k_l z) \tag{4.29}$$

Therefore,

$$\hat{E}_x(z) = \frac{1}{i}\sqrt{\frac{\hbar\omega_l}{\varepsilon V_0}}\left(\hat{a}^\dagger - \hat{a}\right)\sin(k_l z) \tag{4.30}$$

Notice that in the above, $\hat{E}_0$, and $\hat{E}_x(z)$ are all Hermitian operators and they correspond to quantum observables that have randomness associated with them. Also, the operators are independent of time because they are in the Schrodinger picture. The derivation in the Heisenberg picture can be repeated.

In the Schrodinger picture, to get time dependence fields, one has to take the expectation value of the operators with respect to time-varying quantum state vector like the time-varying coherent state.

To let $\hat{E}_x$ have any meaning, it should act on a quantum state. For example,

$$|\psi_E\rangle = \hat{E}_x|\psi\rangle \tag{4.31}$$

Notice that thus far, all the operators derived are independent of time. To derive time dependence of these operators, one needs to find their expectation value with respect to time-dependent state vectors.[9]

To illustrate this, we can take expectation value of the quantum operator $\hat{E}_x(z)$ with respect to a time dependent state vector, like the time-dependent coherent state, Thus

$$\langle E_x(z,t)\rangle = \langle\alpha,t|\hat{E}_x(z)|\alpha,t\rangle = \frac{1}{i}\sqrt{\frac{\hbar\omega_l}{\varepsilon V_0}}\langle\alpha,t|\hat{a}^\dagger - \hat{a}|\alpha,t\rangle$$
$$= \frac{1}{i}\sqrt{\frac{\hbar\omega_l}{\varepsilon V_0}}\left(\tilde{\alpha}^*(t) - \tilde{\alpha}(t)\right)\langle\alpha,t|\alpha,t\rangle = -2\sqrt{\frac{\hbar\omega_l}{\varepsilon V_0}}\Im m(\tilde{\alpha}) \tag{4.32}$$

[9]This is known as the Schrodinger picture.

Using the time-dependent $\tilde{\alpha}(t) = \alpha e^{-i\omega_l t} = |\alpha| e^{-i(\omega_l t + \psi)}$ in the above, we have

$$\langle E_x(z,t)\rangle = 2\sqrt{\frac{\hbar\omega_l}{\varepsilon V_0}}|\alpha|\sin(\omega_l t + \psi) \tag{4.33}$$

where $\tilde{\alpha}(t) = \alpha e^{-i\omega_l t}$. The expectation value of the operator with respect to a time-varying quantum state in fact gives rise to a time-varying quantity.

The above, which is the average of a random field, resembles a classical field. But since it is rooted in a random variable, it has a standard deviation in addition to having a mean.

We can also show that

$$\hat{B}_y(z) = k_l \hat{A}_0 \cos(k_l z) = \sqrt{\frac{2\mu\hbar\omega_l}{V_0}}\hat{\xi} = \sqrt{\frac{\mu\hbar\omega_l}{V_0}}(\hat{a}^\dagger + \hat{a}) \tag{4.34}$$

Again, these are time-independent operators in the Schrodinger picture. To get time-dependent quantities, we have to take the expectation value of the above operator with respect to to a time-varying quantum state.